Raabe's Test :- The Series $\sum u_n$ of positive terms is convergent or divergent according as

$$\lim \left\{ n\left(\frac{u_n}{u_{n+1}} - 1\right)\right\} > 1 \text{ or } < 1$$

OR $\sum u_n$ is convergent if $\lim \left\{ n\left(\frac{u_n}{u_{n+1}} - 1\right)\right\} > 1$

and $\sum u_n$ is divergent if $\lim \left\{ n\left(\frac{u_n}{u_{n+1}} - 1\right)\right\} < 1$

If however, $\lim \left\{ n\left(\frac{u_n}{u_{n+1}} - 1\right)\right\} = 1$, the test fails.

Ex Test for convergence of series

$$1 + \frac{1}{2}\cdot\frac{x^2}{4} + \frac{1.3.5}{2.4.6.8}\cdot\frac{x^4}{8} + \frac{1.3.5.7.9}{2.4.6.8.10}\cdot\frac{x^6}{12} + \cdots$$

Soln:- Here,

$$u_n = \frac{1.3.5 \cdots (4n-7)}{2.4.6 \cdots (4n-6)}\cdot\frac{x^{2n-2}}{4n-4}$$

and,

$$u_{n+1} = \frac{1.3.5 \cdots (4n-7)(4n-5)(4n-3)}{2.4.6 \cdots (4n-6)(4n-4)(4n-2)}\cdot\frac{x^{2n}}{4n}$$

Now,

$$\frac{u_n}{u_{n+1}} = \frac{\cancel{(4n-4)}(4n-2)}{(4n-5)(4n-3)}\cdot\frac{4n}{\cancel{(4n-4)}}\cdot\frac{1}{x^2}$$

$$= \frac{16n^2 - 8n}{16n^2 - 12n - 20n + 15}\cdot\frac{1}{x^2}$$

$$= \frac{16n^2 - 8n}{16n^2 - 32n + 15}\cdot\frac{1}{x^2}$$

$$\therefore \lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \frac{1}{x^2}$$

Hence by D'Alembert ratio test,

the series is convergent if $\frac{1}{x^2} > 1$ or $x^2 < 1$

" " " divergent if $\frac{1}{x^2} < 1$ or $x^2 > 1$.

If $x^2 = 1$ the test fails and apply Raabe's Test.

Ca...

$$\lim_{n\to\infty} n\left\{\frac{4n}{4n+1}-1\right\} = \lim_{n\to\infty} n\cdot\left\{\frac{16n^2-8n}{16n^2-32n+15}-1\right\}$$

$$= \lim_{n\to\infty} n\cdot\left\{\frac{\cancel{16n^2}-8n-\cancel{16n^2}+32n-15}{16n^2-32n+15}\right\}$$

$$= \lim_{n\to\infty}\left\{\frac{24n^2-15n}{16n^2-32n+15}\right\}$$

~~Dividing by n^2~~

$$= \lim_{n\to\infty}\frac{24-\frac{15}{n}}{16-\frac{32}{n}+\frac{15}{n^2}} = \frac{24-0}{16-0+0} = \frac{\cancel{24}\,3}{\cancel{16}\,2} = \frac{3}{2} > 1$$

Hence, the Series is Convergent when $x^2 \leq 1$
and " " " divergent when $x^2 > 1$